Nonlinear estimation

Now the system can be denoted as

\[ x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \]
\[ z_k = h(x_k) + v_k \]

where \( f \) and \( h \) are not necessarily linear functions.

Nonlinear versions of Kalman filter

Kalman filter is the MMSE filter for linear system. While the system is nonlinear, some modifications have been made for Kalman filter to be applied in the nonlinear state estimation.

- Extended Kalman filter (EKF)

EKF linearizes the nonlinear functions around the current estimate using the first order of Taylor expansion. Therefore, the state transition matrix \( F_k \) and observation matrix \( H_k \) in the linear-system case is now defined by the following Jacobians:

\[
F_{k-1} = \left. \frac{\partial f}{\partial x} \right|_{x_{k-1}(k-1), u_{k-1}} \\
H_k = \left. \frac{\partial h}{\partial x} \right|_{x_{k-1}}
\]

Simple as it is, EKF has some problems: 1) If the initial estimate of the state is wrong, or the process is modeled incorrectly, the filter may diverge quickly, due to its linearization. 2) It is not an optimal estimator, and performs poorly in highly nonlinear system. 3) The estimated covariance matrix tends to underestimate the true covariance matrix and therefore risks becoming inconsistent in the statistical sense without addition of “stabilizing noise”.

However, in many practical applications, especially in navigation systems and GPS, EKF works well.


- Unscented Kalman filter (UKF)

UKF uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points (called sigma points) around the mean. These sigma points are then propagated through the nonlinear functions, from which the mean and covariance of the estimate are then recovered.

The advantages of UKF over EKF are that: 1) It captures the true mean and covariance more accurately. 2) There is no need to calculate the Jacobian.


- Ensemble Kalman filter (EnKF)
EnKF is a Monte Carlo approximation of the Kalman filter. It is used for problems with a large number of variables. Since maintaining the covariance matrix is not feasible computationally for high-dimensional system, EnKF tries to represent the distribution of the system state using a collection of state vectors (called an ensemble), and replaces the covariance matrix by the sample covariance computed from the ensemble. Then advancing the pdf in time is achieved by simply advancing each member of the ensemble.

It should be noticed that EnKF relies on the assumption that all probability distributions involved are Gaussian, which is only valid for linear system. However, it is used in practice for nonlinear problems.


Particle filter

The main idea of particle filtering is to approximate the Bayesian estimate by approximating the posterior distribution using a set of samples (called particles).

Consider the system:

\[
\begin{align*}
    x_k &= f(x_{k-1}) + w_k \\
    y_k &= h(x_k) + v_k
\end{align*}
\]

where \( w_k \) and \( v_k \) are i.i.d. sequence with known pdfs, \( f \) and \( h \) are known functions.

We know that the MMSE estimate is the conditional mean. Without loss of generality, we can assume \( w_k \) is zero mean. Then

\[
\hat{x}_{k+1} = E\{x_{k+1}|y_0, y_1, ..., y_k\}
\]

In particle filtering, the expectation with respect to the filtering distribution is approximated using \( P \) samples by

\[
\int f(x_k)p(x_k|y_0, y_1, ..., y_k)dx_k \approx \frac{1}{P} \sum_{L=1}^{P} f\left(x_k^{(L)}\right), \ L \in \{1, 2, ..., P\}
\]

With enough particles, particle filters can make a much more accurate approximation than EKF or UKF.

There are many algorithms for particle filtering. Refer to http://en.wikipedia.org/wiki/Particle_filter for detail.