Convergence

1. Convergence of Sequence of numbers

1) Definition ($\epsilon - N$ game):
A sequence of numbers $\{r_n\}$ converges to the number $r$ ($r_n \to r$), iff
$$\forall \epsilon > 0, \exists N(\epsilon), \text{ s.t. } \forall n > N(\epsilon), |r_n - r| < \epsilon$$

2) Cauchy sequence:
The sequence $\{r_n\}$ is Cauchy iff
$$\forall \epsilon > 0, \exists N(\epsilon), \text{ s.t. } \forall n, m > N(\epsilon), |r_n - r_m| < \epsilon$$

3) Complete space
A metric space $M$ is said to be complete (or Cauchy) if every Cauchy sequence of points in $M$ has a limit that is also in $M$, or alternatively if every Cauchy sequence in $M$ converges in $M$.

2. Convergence of sequence of functions

1) Point wise convergence:
The sequence of functions $\{f_n(x)\}$ converges at $x = x_0$ to $f(x_0)$ iff
$$\forall \epsilon > 0, \exists N(\epsilon), \text{ s.t. } \forall n > N(\epsilon), |f_n(x_0) - f(x_0)| < \epsilon$$

Note:
- $N$ depends both on $\epsilon$ and $x_0$!
- If convergence occurs at point $x = x_0$, we write $f_n(x_0) \to f(x_0)$; If convergence occurs for every $x_0 \in X$, we simply write $f_n \to f$.

2) Uniform convergence
If in the above definition, $N$ depends only on $\epsilon$ but not on $x_0$, the convergence is said to be uniform.

Note:
- Uniform convergence means that the same $N$ works for every point $x \in X$ in the $\epsilon - N$ game. Or we can say that the sequence of functions $\{f_n(x)\}$ has the same rate of convergence at every point.
- Uniform convergence is a condition that allows interchanging integration with the lim operation.
- Convergence occurring for every $x_0 \in X$ does not imply uniform convergence!

3. Convergence of sequence of random variables

1) Related definitions
• Bad set: $B_n^\epsilon = \{ \omega : |x_n(\omega) - x(\omega)| > \epsilon \}$

• Supremum (sup) or least upper bound (lub): The supremum of a set of numbers is the smallest number which is greater than or equal to each number in the set.

2) Convergence wp1

The sequence of random variables $\{x_n(\omega)\}$ converges wp1 to the random variable $x(\omega)$ iff

$$\forall \epsilon, \delta > 0, \exists N, \text{ s.t. } \mathbb{P}\{\bigcup_{n \geq N} B_n^\epsilon\} < \delta$$

• Equivalent statement:

$$\forall \epsilon, \delta > 0, \exists N, \text{ s.t. } \lim_{n \to \infty} \mathbb{P}\{x_n(\omega) \to x(\omega)\} = 1$$

Or

$$\forall \epsilon, \delta > 0, \exists N, \text{ s.t. } \mathbb{P}\{\omega : \sup_{n \geq N} |x_n(\omega) - x(\omega)| > \epsilon\} < \delta$$

• Denotation: with probability one (wp1), almost sure (a.s.), almost everywhere (a.e.), p-almost sure (p-a.s.), p-almost everywhere (p-a.e.).

3) Convergence in Mean Square Sense

The sequence of random variables $\{x_n\}$ in $\mathcal{H}$ converges in mss to the random variable $x$ iff

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

Or

$$\lim_{n \to \infty} \mathbb{E}[|x_n(\omega) - x(\omega)|^2] = 0$$

• It allows us to interchange expectation with the limit taken in mean square sense. E.g.,

$$\lim_{n \to \infty} \mathbb{E}[x_n] = \mathbb{E}[\text{l.i.m. } x_n]$$

• Convergence in pth-mean: \(\lim_{n \to \infty} [\mathbb{E}[|x_n(\omega) - x(\omega)|^p]]^{1/p} = 0\)

4) Convergence in probability

The sequence of random variables $\{x_n(\omega)\}$ converges in probability to the random variable $x(\omega)$ iff

$$\forall \epsilon, \delta > 0, \exists N, \text{ s.t. } \forall n \geq N, \mathbb{P}\{B_n^\epsilon\} < \delta$$

or $\forall \epsilon > 0 : \lim_{n \to \infty} \mathbb{P}\{B_n^\epsilon\} = 0$.

5) Convergence in distribution

A sequence of random variables $x_n$ converges in distribution to the random variable $x$ iff the sequence of cumulative distribution functions converges to $F_x$ at every point of continuity of $F_x$:

$$\lim_{n \to \infty} F_{x_n}(X) = F_x(X)$$
6) Relations between types of convergence

- *convergence at everywhere* ⇒ *wp1 convergence*
- *wp1 convergence* ⇒ *p – lim convergence*
- *mss convergence* ⇒ *p – lim convergence*
- *p – lim convergence* ⇒ *convergence in probability*
- The relation between *wp1 convergence* and *mss convergence* should be stated with additional conditions
- For every type of convergence, if, for every point where convergence occurs, the rate of convergence is the same, then the convergence is *uniform convergence*.